Size-dependent pull-in analysis of electrically actuated micro-plates based on the modified couple stress theory

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Abstract. In view of the difficulties arisen in determining the solutions of linear eigenvalue problems of fully-clamped rectangular micro-plates, the size-dependent pull-in analysis of such systems has not been carried out yet. Therefore, size-dependent pull-in analysis of such systems is the main goal of the present work. To this end, the size-dependent equation of equilibrium for an electrically actuated fully-clamped rectangular micro-plate based on the combination of the modified couple stress theory and the Kirchhoff plate model is utilized and solved through the computationally efficient single mode Galerkin reduced order method. The linear and un-damped mode-shapes of the system, which are obtained by the extended Kantorovich method (EKM), are employed as the weight functions in the Galerkin procedure. The findings of the present work are compared and validated against the available results in the literature and excellent agreements between them are observed. The results show that considering the effect of couple stress components increases the instability threshold of the system.

Introduction

Thanks to recent advances in the technology of micro-electro-mechanical systems (MEMS), micro sensors and actuators are widely used in different applications and industries. Nowadays, MEMS are key components of many devices because of their small size, low power consumption, high reliability and the ability of batch fabrication [1].

MEMS devices are produced in different types such as micro-beams, micro-plates and micro-mirrors. Parallel rectangular micro-plates are the building blocks of plate-type MEMS. These structures consist of two parallel electrodes which act by applying an external voltage. One of these electrodes is stationary and considered as a rigid substrate and the other one is a flexible micro-plate. Electrostatic force made by applying the external voltage, forces the flexible electrode to deflect toward the rigid substrate. In this manner the mechanical elastic restoring force is the only source which resists against the electrostatic attraction. It is noteworthy that the applied voltage has an upper limit in which the elastic restoring force cannot overcome the electrostatic attraction and the structure suddenly collapses toward the rigid substrate. This phenomenon is called pull-in instability and the minimum associated voltage with that is called pull-in voltage. Pull-in instability was observed by Nathanson et al. [2] for the first time.

Recent experiments show that the mechanical behaviors of materials in micron scales are size-dependent [3]. Size-dependency is an inherent property of the certain materials which emerges when the characteristic size of the structure (e.g. the diameter or the thickness) is comparable to the material length scale parameter. Material length scale parameter is one of the mechanical properties and can be determined by some typical experiments such as micro-torsion test [3] and micro-bend test [4].

The common classical theory (CT) of continuum mechanics cannot predict the size-dependent behavior of materials in micron and sub-micron scales. Therefore, the size-dependent theories have been developed. These theories contain some additional material constants besides two classical Lame's constants for isotropic materials: The classical couple stress theory (CCST), the classical strain gradient theory (CSGT), the modified couple stress theory (MCST) and the strain gradient (SGT) theory of elasticity include two, five, one and three additional material constants, respectively [3, 5-7]. Many researches have been done to investigate the size-dependent behavior of the materials and a lot of them are concentrated on micro-beams while there are a few researches done on micro-plates especially in the case of using MCST. Tsiatas [8] presented a size-dependent model for thin plates based on the Kirchhoff's assumptions. He solved the boundary value problem using the method of fundamental solutions (MFS) which is a boundary-type meshless method. He showed that the deflection of the micro-plates decreases nonlinearly with the increase of material length scale parameter and the MCST to CT ratio of the deflection is influenced only by the Poisson's ratio of the micro-plate. Jomehzadeh et al. [9] investigated the natural frequencies of thin micro-plates with both circular and rectangular shapes. However, their research was limited to rectangular micro-plates with two opposite edges simply supported. They found that considering the size effect increases the natural frequencies of the system. Roque et al. [10] used the MCST together with the first-order shear deformation plate model to study the size-dependent bending of simply supported isotropic micro-plates. They solved the equilibrium equations numerically through a meshless method. Ke et al. [11] investigated the size-dependent bending, buckling and free vibrations of annular Mindlin micro-plates made of functionally graded materials (FGMs) based on the MCST. They solved their equations using differential quadrature method. They found that when the thickness is much greater than the length scale parameter in micro-plates, the size effect is negligible. Ke et al. [12] also developed a similar model for the free vibration analysis of rectangular Mindlin's micro-plates and found that the size effect is significant when the thickness of micro-plate is close to its material length scale parameter. They utilized the p-version of the Ritz method to solve the governing equations of motion. Thai and Choi [13] presented size-dependent models for bending, buckling, and free vibrations of non-linear functionally graded Kirchhoff's and Mindlin's simply supported micro-plates based on the MCST. Although, they could not solve their governing equations for non-linear FG plates and they presented some analytical solutions for other cases of simply supported micro-plates. Zhang et al. [14] presented a novel size-dependent plate element including 4 nodes and 15 degrees of freedom for each node based on a framework of the MCST and the Mindlin's plate model for analyzing the static bending, free vibration and buckling behaviors of thick rectangular micro-plates.

Size-dependent pull-in analysis of plate-type MEMS has not been carried out yet in the open literature. The main aim of this study is to present a size-dependent pull-in analysis for a fully-clamped rectangular micro-plate using the MCST. To this end, the governing partial differential equation (PDE) of equilibrium is investigated and reduced to an algebraic equation through the Galerkin weighted residual method. The linear and un-damped mode-shapes of the micro-plate, which are obtained by the EKM, are utilized as the admissible basis functions. It is found that using only the first mode not only can provide high accurate results in comparison with available empirical observations in the literature, but also it can remove the small gap between the findings of CT and experiments [15]. A detailed parametric study is also conducted to illustrate the significant effects of couple stress components on the pull-in instability threshold of rectangular micro-plates. It is found that the ratio of MCST to CT pull-in voltages is independent of the plate aspect ratio and only depends on Poisson's ratio for systems without the effect of axial residual stresses.

Size-dependent mathematical modeling

Consider two parallel rectangular electrodes of plate-type MEMS. One of them is an isotropic fully-clamped rectangular micro-plate and the other one is a stationary rigid substrate. The length, width, thickness, density, Young's modulus of elasticity, and Poisson's ratio of the flexible electrode are a, b, h, ρ, E, and v respectively. The initial gap between two electrodes is d. Also X, Y, and Z are the respective coordinates along the length, width, and thickness, and W is the deflection. It is to be noted that, for convenience, the coordinate system is attached at the center of the mid-plane of the micro-plate. According to the MCST presented by Yang et al. [7], the strain energy density consists of both the strain and curvature components which are conjugated with the classical stress and couple stress components, respectively. Also the MCST includes only one additional constant parameter known as the material length scale parameter. Considering the MCST and

Kirchhoff plate model, the governing equation of equilibrium for an electrically actuated rectangular micro-plate can be written as [8, 16]:

$$\left(\frac{Eh^3}{12\left(1-v^2\right)}+\frac{Ehl^2}{2\left(1+v\right)}\right)\left(\frac{\partial^4 W}{\partial X^4}+2\frac{\partial^4 W}{\partial X^2\partial Y^2}+\frac{\partial^4 W}{\partial Y^4}\right)-\left(\hat{N}_x\frac{\partial^2 W}{\partial X^2}+\hat{N}_y\frac{\partial^2 W}{\partial Y^2}\right)=\frac{\varepsilon V^2}{2\left(d-W\right)^2} \tag{1}$$

where l is the material length scale parameter, V is the polarized DC voltage and $\hat{N}_x$ and $\hat{N}_y$ are the axial residual forces in the X and Y directions, respectively. To non-dimensionalize the governing PDE, dimensionless parameters $x = X/a$, $y = Y/b$, and $w = W/d$ are introduced. By substituting these parameters into Eq. 1, the non-dimensionalized form of the governing equation takes the form:

$$\left(1+6\alpha_2\right)\left(\frac{\partial^4 w}{\partial x^4}+2\alpha_1^{\,2}\frac{\partial^4 w}{\partial x^2\partial y^2}+\alpha_1^{\,4}\frac{\partial^4 w}{\partial y^4}\right)-\left(N_x\frac{\partial^2 w}{\partial x^2}+\alpha_1^{\,4}N_y\frac{\partial^2 w}{\partial y^2}\right)=\frac{\beta}{(1-w)^2} \tag{2}$$

where

$$\alpha_1=\frac{a}{b},\ \alpha_2=\frac{1-v}{\left(h/l\right)^2},\ \beta=\frac{\varepsilon a^4 V^2}{2Dd^3},\ D=\frac{Eh^3}{12(1-v^2)},\ N_x=\frac{a^2\hat{N}_x}{D},\ N_y=\frac{b^2\hat{N}_y}{D} \tag{3}$$

Solution procedure

According to the Galerkin weighted residual method, the static deflection of the micro-plate can be expressed as [1, 17]:

$$w\left(x,y\right)=\sum_{i=1}^{n}\varphi_i\left(x,y\right)u_i \tag{4}$$

where n is the number of degrees of freedom, φ_i is the ith linear and un-damped mode-shape of the micro-plate, and u_i is the ith un-known generalized coordinate which should be determined. It is proved that, for micro-plates' problems, utilizing only the first linear and un-damped mode-shape for the transverse deflection can provide sufficient accuracy [18, 19]. Hence, the solution is expressed as:

$$w\left(x,y\right)=\varphi_{11}\left(x,y\right)u \tag{5}$$

Substituting Eq. 5 into Eq. 2 and applying the Galerkin weighted residual method [17], one can obtain:

$$\begin{aligned}&\left[\left(1+6\alpha_2\right)\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\left(\frac{\partial^4\varphi_{11}}{\partial x^4}+2\alpha_1^{\,2}\frac{\partial^4\varphi_{11}}{\partial x^2\partial y^2}+\alpha_1^{\,4}\frac{\partial^4\varphi_{11}}{\partial y^4}\right)\varphi_{11}dxdy\right.\\&\left.-\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\left(N_x\frac{\partial^2\varphi_{11}}{\partial x^2}+\alpha_1^{\,4}N_y\frac{\partial^2\varphi_{11}}{\partial y^2}\right)\varphi_{11}dxdy\right]u=\beta\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\frac{\varphi_{11}}{\left(1-u\varphi_{11}\right)^2}dxdy\end{aligned} \tag{6}$$

In Eq. 6, the first linear and un-damped mode-shape of the plate (i.e. φ_{11}) is obtained through the EKM. To do so, the mode-shape of the rectangular micro-plate is considered as a multiplication of two separable functions as [19, 20]:

$$\varphi_{11}\left(x,y\right)=f\left(x\right)g\left(y\right) \tag{7}$$

According to the iterative procedure of the EKM [19, 20], the variational form of the linear and un-damped eigenvalue problem associated with the present micro-plate problem is considered. At the first iteration step, one of the aforementioned separable functions (for example $g(y)$) is considered as a known guess function such that it satisfies all corresponding separated boundary conditions [19]. By choosing $g(y)=\left(4y^2-1\right)^2$ and substituting this function into the variational form of the present eigenvalue PDE and employing the fundamental lemma of variational calculus [17], the other separable function (i.e. $f(x)$) will be determined through solving a resultant linear ordinary differential equation (ODE) with its corresponding separated boundary conditions [19]. At the next step of iteration, the determined $f(x)$ is employed as the known function and import into the variational form of the eigenvalue PDE again. Utilizing the mentioned procedure $g(y)$ can be obtained by solving another ODE with its corresponding separated boundary conditions. This procedure should be continued till the convergence is achieved. It is proved that only one iteration yields the mode-shape of a fully clamped rectangular micro-plate accurately [19]. Following the EKM procedure, the eigenvalue PDE will be separated to two following ODEs as [19, 20]:

$$\frac{d^4f}{dx^4}-I_1\frac{d^2f}{dx^2}+\left(I_2-\bar{\alpha}_2\omega_{mn}^2\right)f=0 \tag{8a}$$

$$\frac{d^4g}{dy^4}-I_1'\frac{d^2g}{dy^2}+\left(I_2'-\bar{\alpha}_2'\omega_{mn}^2\right)g=0 \tag{8b}$$

where

$$I_1=\frac{2\alpha_1^2\left(1+6\alpha_2\right)\int_{-1/2}^{1/2}\left(\frac{dg}{dy}\right)^2dy+N_x\int_{-1/2}^{1/2}g^2\,dy}{\left(1+6\alpha_2\right)\int_{-1/2}^{1/2}g^2\,dy},\ I_2=\frac{\alpha_1^4\left[\left(1+6\alpha_2\right)\int_{-1/2}^{1/2}\left(\frac{d^2g}{dy^2}\right)^2dy+N_y\int_{-1/2}^{1/2}\left(\frac{dg}{dy}\right)^2dy\right]}{\left(1+6\alpha_2\right)\int_{-1/2}^{1/2}g^2\,dy}$$

$$I_1'=\frac{2\left(1+6\alpha_2\right)\int_{-1/2}^{1/2}\left(\frac{df}{dx}\right)^2dx+\alpha_1^2N_y\int_{-1/2}^{1/2}f^2\,dx}{\alpha_1^2\left(1+6\alpha_2\right)\int_{-1/2}^{1/2}f^2\,dx},\ I_2'=\frac{\left(1+6\alpha_2\right)\int_{-1/2}^{1/2}\left(\frac{d^2f}{dx^2}\right)^2dx+N_x\int_{-1/2}^{1/2}\left(\frac{df}{dx}\right)^2dx}{\alpha_1^4\left(1+6\alpha_2\right)\int_{-1/2}^{1/2}f^2\,dx} \tag{9}$$

$$\bar{\alpha}_2=\frac{1}{1+6\alpha_2},\ \bar{\alpha}_2'=\frac{1}{\alpha_1^4\left(1+6\alpha_2\right)}$$

The mode-shapes of a fully-clamped rectangular micro-plate will be obtained by applying the separated boundary conditions on the solutions of Eqs. 8. By substituting the first determined linear and un-damped mode-shape of the plate, the only unknown of the non-linear algebraic Eq. 6 can be obtained iteratively. To do so, the non-linear Eq. 6 is linearized using the zeroth order Tailor's expansion of its non-linear terms about the unknown generalized coordinate u and the following iterative solution is obtained:

$$u_{N+1}=\frac{\beta\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\frac{\varphi_{11}}{(1-(u_N\varphi_{11}))^2}dxdy}{\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\left(\left(1+6\alpha_2\right)\left(\frac{\partial^4\varphi_{11}}{\partial x^4}+2\alpha_1^2\frac{\partial^4\varphi_{11}}{\partial x^2\partial y^2}+\alpha_1^4\frac{\partial^4\varphi_{11}}{\partial y^4}\right)-\left(N_x\frac{\partial^2\varphi_{11}}{\partial x^2}+\alpha_1^4N_y\frac{\partial^2\varphi_{11}}{\partial y^2}\right)\right)\varphi_{11}dxdy} \tag{10}$$

At the first step, the value of the unknown parameter u (i.e. u_0) is assumed to be zero. This iterative procedure will be continued till the convergence is occurred or pull-in is happened. The convergence criteria and pull-in condition are set to $\left(u_{N+1}-u_N\right)/u_N\le 10^{-6}$ and $w_{\text{mid-point}}=\varphi_{11}\left(0,0\right)u_{N+1}\ge 1$, respectively [19, 21].

Results and discussions

In order to verify the present study, the results are validated by the experimental observations of Francais and Dufour [15]. In this case the axial residual stresses are neglected and the specifications of the system are given in Table 1.

Table 1. Material and geometrical parameters of the system

p	E	v	h	d	l [22]
$2320\,\mathrm{kg/m^3}$	$169\,\mathrm{GPa}$	0.3	$20\,\mu\mathrm{m}$	$5\,\mu\mathrm{m}$	$0.592\,\mu\mathrm{m}$

According to Fig. 1 the results are in excellent agreement with those obtained by Francais and Dufour [15]. To find the results of the CT using the present procedure the value of the material length scale parameter is set to zero (i.e. $l=0$). Fig. 1 also shows the comparison between the MCST and CT results. It is seen that use of the MCST increases the pull-in voltage and makes the results more accurate and compatible with the empirical observations.

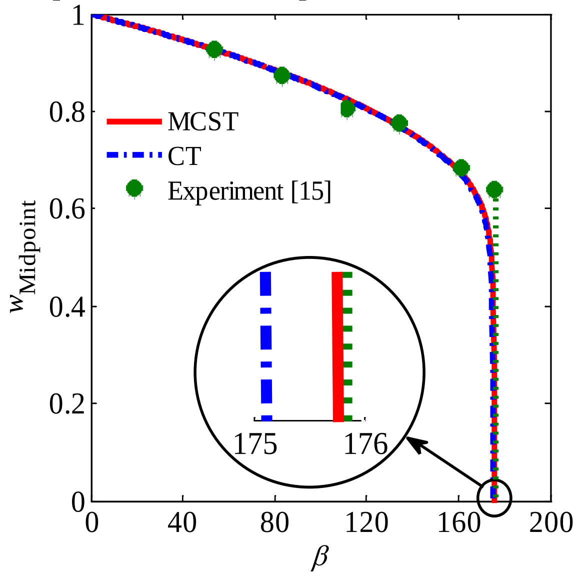

Fig. 1. Non-dimensional midpoint deflection versus non-dimensional applied voltage

According to Eq. 3 the coefficient of the MCST (i.e. α_2) is a function of h/l. On the other hand, to investigate the effect of using the MCST in compare to the CT the ratio of the pull-in voltage in the both theories is considered (i. e. $\beta_{\mathrm{PI}}^{\mathrm{MCST}}/\beta_{\mathrm{PI}}^{\mathrm{CT}}$). Fig. 2 represents $\beta_{\mathrm{PI}}^{\mathrm{MCST}}/\beta_{\mathrm{PI}}^{\mathrm{CT}}$ versus h/l for different values of aspect ratio when there exists no axial load.

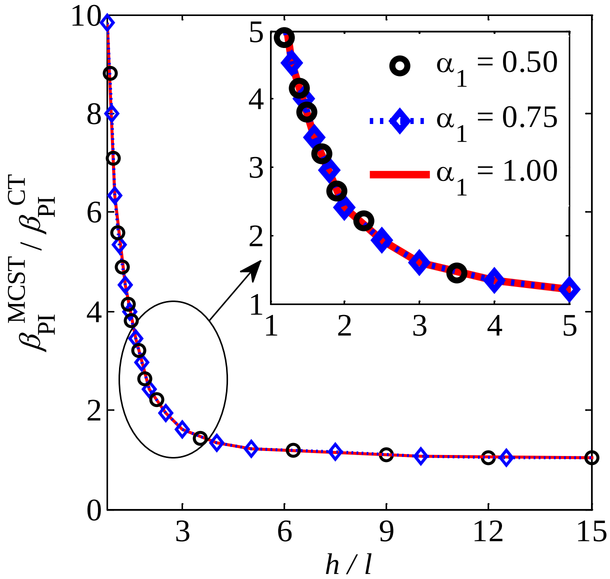

Fig. 2. Effect of plate's aspect ratio on $\beta_{\mathrm{PI}}^{\mathrm{MCST}}/\beta_{\mathrm{PI}}^{\mathrm{CT}}$ ($N_x=N_y=0$)

As it is seen in Fig. 2, when there exists no axial load, the ratio of the pull-in voltage found by both of the theories is independent of the plate's aspect ratio. Fig. 3 shows the ratio of $V_{PI}^{MCST}/V_{PI}^{CT}$ considering the axial loads. It is to be noted that the ratio of $V_{PI}^{MCST}/V_{PI}^{CT}$ can be calculated as $\sqrt{\beta_{PI}{}^{MCST}/\beta_{PI}{}^{CT}}$. As It is observed form Fig.3, the ratio of $V_{PI}^{MCST}/V_{PI}^{CT}$ increases by applying compressive axial load and decreases when tensile axial loads are applied. Also the effect of compressive axial load is more than that of tensile one. In this case the ratio of $V_{PI}^{MCST}/V_{PI}^{CT}$ decreases by reducing the plate's aspect ratio.

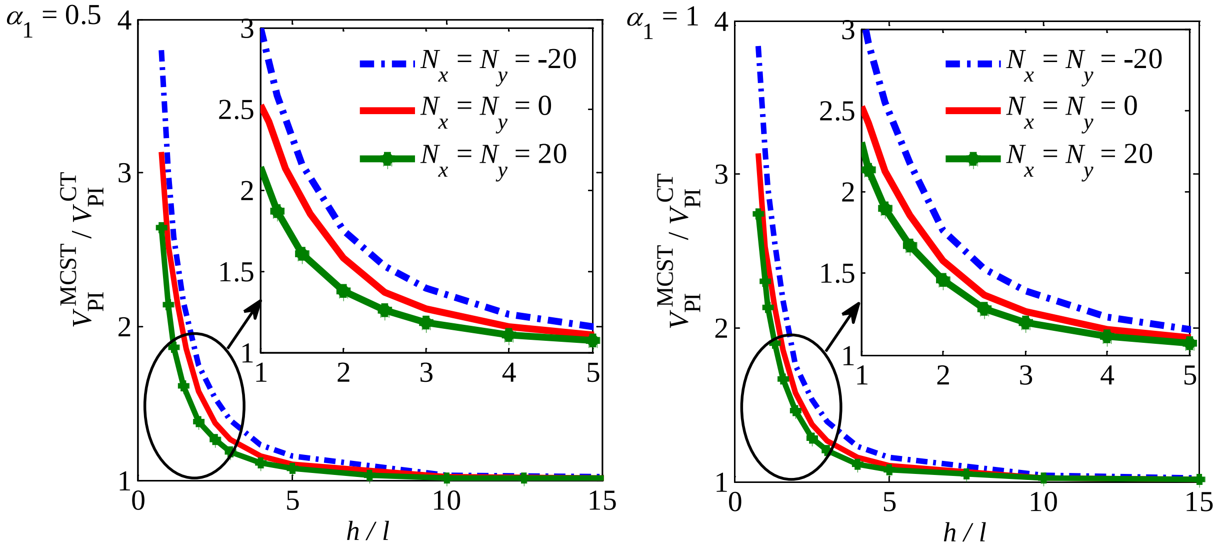

Fig. 3. Effects of plate's aspect ratio and applied axial loads on the ratio of $V_{PI}^{MCST}/V_{PI}^{CT}$

Summary

In this study, a combination of the Galerkin projection method and the EKM has been successfully utilized to investigate the size-dependent pull-in instability of fully clamped rectangular micro-plates based on the MCST. The main conclusions of the present work can be summarized to:

- The decrease of the size effect parameter (i.e. h/l) increases the difference between the results of the MCST and CT.
- The differences between the MCST and CT results for pull-in voltage, are usually negligible when $h/l \geq 10$. However, this range will be extended more for axially compressed micro-plates.
- The ratio of $V_{PI}^{MCST}/V_{PI}^{CT}$ is independent of the plate's aspect ratio for axially unstressed micro-plates.
- The ratio of $V_{PI}^{MCST}/V_{PI}^{CT}$ decreases non-linearly by a decrease of plate's aspect ratio for axially stressed micro-plates.

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