Multiple pattern-dependent controller design for Markovian jump linear systems

Taesoo Kim and Sung Hyun Kim*

Abstract—This paper is concerned with deriving the stabilization condition for discrete-time Markovian jump linear systems (MJLSs) with multiple patterns of mode transition probabilities. In the derivation, a method of establishing the pattern-dependent transition probability matrices is proposed, which offers possibilities for extending our result to other issues of MJLSs.

Keywords—Markovian jump systems, control synthesis, networked control systems

I. Introduction

OVER the past decades, considerable efforts have been made in the study of Markovian jump linear systems (MJLSs) because a class of dynamic systems subject to random abrupt variations can be modeled by MJLSs (see [1], [2] and the references therein). Based on such efforts, the MJLS model has been applied in many practical applications [3], [4]. However, despite the numerous works available, most studies in the available literature regarding the control synthesis problem were found to have been carried out without consideration of the multiple patterns for mode transition probabilities.

Indeed, as reported in [5], the use of unified pattern-oriented transition probabilities may pose considerable uncertainties in the process of expressing the considered systems as MJLSs. For this reason, [5] proposed a method capable of stabilizing multiple pattern-dependent MJLSs in order to improve the convergence rate of the state response of networked control systems (NCSs). However, the drawback of [5] lies in the fact that the NCSs are designed irrespective of the utilization of the sequence that indicates the variation of patterns.

Motivated by the above concern, this paper focuses on deriving the stabilization condition for a class of discrete-time MJLSs with multiple patterns for mode transition probabilities. In contrast with existing results, this paper employs an additional discrete-time Markov process to incorporate information related to patterns into the derivation of the stabilization conditions. In addition, this paper proposes a method of establishing the pattern-dependent transition probability matrices, which offers possibilities for extending our result to other issues of MJLSs.

II. Notation

The notations $X \geq Y$ and $X > Y$ indicate that $X - Y$ is positive semi-definite and positive definite, respectively. In symmetric block matrices, $(*)$ is used as an ellipsis for terms that are induced by symmetry. For any square matrix $\mathcal{Q}$, $\mathbf{He}(\mathcal{Q}) = \mathcal{Q} + \mathcal{Q}^T$ and $\mathbf{diag}(e_1, e_2, \cdots, e_n)$ indicates a diagonal matrix with diagonal entries $e_1, e_2, \cdots, e_n$. For $a_i \in \mathcal{N}^+ \triangleq \{1, 2, \cdots\}$ such that $a_i < a_{i+1}$, $i \in \mathcal{N}_n^+ \triangleq \{1, 2, \cdots, n\}$, the notation

$$[\mathcal{Q}_i]_{i\in\{a_1,\cdots,a_n\}} = [\mathcal{Q}_{a_1} \cdots \mathcal{Q}_{a_n}]$$
$$[\mathcal{Q}_{ij}]_{i,j\in\{a_1,\cdots,a_n\}} = \Big[[\mathcal{Q}_{a_1 j}]^T_{j\in\{a_1,\cdots,a_n\}} \ \cdots [\mathcal{Q}_{a_n j}]^T_{j\in\{a_1,\cdots,a_n\}}\Big]^T,$$

where $\mathcal{Q}_i$ and $\mathcal{Q}_{ij}$ denote real submatrices with appropriate dimensions. $\mathbf{E}(\cdot)$ denotes the mathematical expectation.

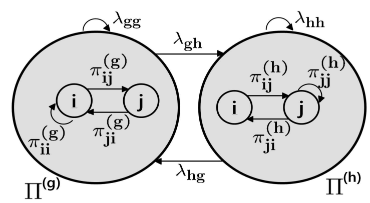

Fig. 1. Transition of multiple patterns for MTP matrices.

III. Preliminaries

Consider the following discrete-time MJLSs:

$$x_{k+1} = A(r_k)x_k + B(r_k)u_k, \tag{1}$$

where $x_k \in \mathcal{R}^{n_x}$ and $u_k \in \mathcal{R}^{n_u}$ denote the state and the control input, respectively; and r_k denotes a discrete-time Markov process on the probability space that takes the values in a finite set $\mathcal{N}_s^+$. Here, we employ an additional discrete-time Markov process $p_k \in \mathcal{N}_c^+$ to describe the multiple patterns for mode transition probabilities (MTPs) whose transition probabilities are given by $\mathbf{Pr}(p_{k+1} = h | p_k = g) = \lambda_{gh}$ (see Fig. 1). Then, the mode transition probabilities of r_k are taken to be $\mathbf{Pr}(r_{k+1} = j | p_k = g, r_k = i) = \pi_{ij}^{(g)}$, where p_0 and r_0 become the initial operation mode and pattern, respectively. In particular, we define the MTP matrix for any pattern g as $\Pi^{(g)} = [\pi_{ij}^{(g)}]_{i,j\in\mathcal{N}_s^+}$ (see Fig. 1).

Now, let us consider the following state-feedback control law: $u_k = F(p_k, r_k)x_k$, where $F(p_k, r_k)$ denotes the pattern-dependent control gain, to be designed later. For later convenience, we set $A_i = A(r_k = i)$, $B_i = B(r_k = i)$, and $F_{gi} = F(p_k = g, r_k = i)$. Then, the closed-loop system is described as follows:

$$x_{k+1} = \bar{A}_{gi}x_k, \tag{2}$$

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where $\bar{A}_{gi} = A_i + B_i F_{gi}$. In addition, the following definition is adopted to address the stabilization problem under consideration.

Definition 3.1 ([2], [5]): System (2) is said to be mean square stable if its solution is such that $\lim_{k\to\infty} \mathbf{E}(||x_k||^2) = 0$ for any initial conditions x_0, p_0, and r_0.

Lemma 3.1: System (2) is said to be mean square stable if there exist matrices $F_{gi} \in \mathcal{R}^{n_u \times n_x}$ and symmetric matrices $P_{gi} \in \mathcal{R}^{n_x \times n_x}$ such that

$$0 > \mathcal{M}_{gi} = \left(\sum_{h=1}^{c} \sum_{j=1}^{s} \lambda_{gh} \pi_{ij}^{(h)} \bar{A}_{gi}^T P_{hj} \bar{A}_{gi} \right) - P_{gi}, \quad \forall g, i. \tag{3}$$

Proof: Consider the following Lyapunov function candidate dependent on both the pattern p_k and the mode r_k: $V_k = V(p_k, r_k) = x_k^T P(p_k, r_k) x_k$, where $P(p_k, r_k) > 0$ for all $p_k \in \mathcal{N}_c^+$ and $r_k \in \mathcal{N}_s^+$. Then, from the Rayleigh quotient, it follows that

$$\mathbf{E}(V_k) \geq \min_{g \in \mathcal{N}_c^+, i \in \mathcal{N}_s^+} \lambda_{\min}(P_{gi}) \cdot \mathbf{E}\left(||x_k||^2\right), \tag{4}$$

where $P_{gi} = P(p_k = g, r_k = i)$ and $\lambda_{\min}(P_{gi})$ denotes the minimum eigenvalue of P_{gi}. Note that there exists a scalar δ such that $0 < \delta I \leq \min_{g \in \mathcal{N}_c^+, i \in \mathcal{N}_s^+} \lambda_{\min}(P_{gi})$ in the sense that $P_{gi} > 0$ for all g, i. As a result, $\mathbf{E}(V_k) \geq \delta \mathbf{E}\left(||x_k||^2\right)$, which leads to

$$\mathbf{E}\left(||x_k||^2\right) \leq \delta^{-1} \mathbf{E}(V_k), \ \delta > 0. \tag{5}$$

Next, we see that

$$\begin{aligned} &\mathbf{E}\left(V(p_{k+1}, r_{k+1} | p_k = g, r_k = i)\right) - V(p_k = g, r_k = i) \\ &= x_k^T \left(\sum_{h=1}^{c} \sum_{j=1}^{s} \lambda_{gh} \pi_{ij}^{(h)} \bar{A}_{gi}^T P_{hj} \bar{A}_{gi} - P_{gi} \right) x_k \\ &= x_k^T \mathcal{M}_{gi} x_k. \end{aligned} \tag{6}$$

In addition, for $x_k \neq 0$,

$$\begin{aligned} &\frac{\mathbf{E}\left(V(p_{k+1}, r_{k+1} | p_k = g, r_k = i)\right) - V(p_k = g, r_k = i)}{V(p_k = g, r_k = i)} \\ &= -\frac{x_k^T(-\mathcal{M}_{gi}) x_k}{x_k^T P_{gi} x_k} \leq -\min_{g,i} \frac{\lambda_{\min}(-\mathcal{M}_{gi})}{\lambda_{\max}(P_{gi})}. \end{aligned} \tag{7}$$

Let $\alpha - 1 = -\min_{g,i} \frac{\lambda_{\min}(-\mathcal{M}_{gi})}{\lambda_{\max}(P_{gi})}$. Then, (3) implies $\alpha < 1$, and (7) allows that

$$\mathbf{E}\left(V(p_{k+1}, r_{k+1} | p_k = g, r_k = i)\right) \leq \alpha V(p_k = g, r_k = i), \quad 0 < \alpha < 1,$$

that is, $\mathbf{E}(V_k) \leq \alpha^k V(p_0, r_0)$ for any x_0, p_0, and r_0. As a result, (5) can be converted into $0 \leq \mathbf{E}\left(||x_k||^2\right) \leq \delta^{-1} \alpha^k V(p_0, r_0)$, where $0 < \alpha < 1$. Hence, we can see that $\lim_{k\to\infty} \mathbf{E}\left(||x_k||^2\right) = 0$ because $\lim_{k\to\infty} \alpha^k = 0$. Therefore, by Definition 1, the proof can be completed. ∎

IV. Main results

For simplicity of the discussion, this paper assumes that the sequence of patterns, designated as PAT in Fig. 2, is generated by a proper pattern indicator. Based on PAT, we can then reconstruct the sequences SEQ 1, SEQ 2, $\cdots$, SEQ c from SEQ 0, which result in the MTP matrices $\Pi^{(1)}$, $\cdots$, $\Pi^{(c)}$ required in Fig. 1.

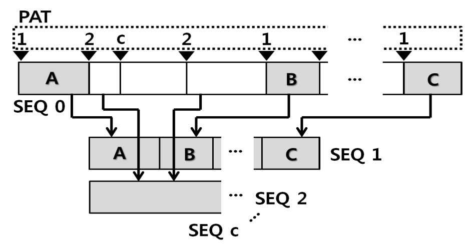

Fig. 2. Diagram for the construction of MTP matrices $\Pi^{(1)}, \cdots, \Pi^{(c)}$.

The following theorem presents a set of conditions for the control synthesis of (2).

Theorem 4.1: Suppose that there exist matrices $\bar{F}_{gi} \in \mathcal{R}^{n_u \times n_x}$ and symmetric matrices $\bar{P}_{gi}$, $Q_{gi,hj} \in \mathcal{R}^{n_x \times n_x}$ such that, for all g, i,

$$0 < \bar{P}_{gi} - \sum_{h=1}^{c} \sum_{j=1}^{s} \lambda_{gh} \pi_{ij}^{(g)} Q_{gi,hj}, \tag{8}$$

$$0 \leq \begin{bmatrix} \bar{P}_{hj} & A_i \bar{P}_{gi} + B_i \bar{F}_{gi} \\ (*) & Q_{gi,hj} \end{bmatrix}, \ \forall h, j. \tag{9}$$

Then, the closed-loop control system (2) is stochastically stable and the mode-dependent control gains are given by $F_{gi} = \bar{F}_{gi} \bar{P}_{gi}^{-1}$ for all g, i.

Proof: By Lemma 1, the stability condition of (2) is given by $0 < P_{gi} - \sum_{h=1}^{c} \sum_{j=1}^{s} \lambda_{gh} \pi_{ij}^{(g)} \bar{A}_{gi}^T P_{hj} \bar{A}_{gi}$. Furthermore, performing a congruent transformation to the stability condition by $\bar{P}_{gi} = P_{gi}^{-1}$ yields

$$0 < \bar{P}_{gi} - \sum_{h=1}^{c} \sum_{j=1}^{s} \lambda_{gh} \pi_{ij}^{(g)} \bar{P}_{gi} \bar{A}_{gi}^T P_{hj} \bar{A}_{gi} \bar{P}_{gi}, \ \forall g, i. \tag{10}$$

In the sense that $\lambda_{gh} \geq 0$ and $\pi_{ij}^{(g)} \geq 0$, (10) can be converted into (8),

$$0 \leq Q_{gi,hj} - \bar{P}_{gi} \bar{A}_{gi}^T P_{hj} \bar{A}_{gi} \bar{P}_{gi}. \tag{11}$$

Finally, after applying the Schur complement to (11), it becomes (9), where $\bar{F}_{gi} = F_{gi} \bar{P}_{gi}$. ∎

V. Numerical examples

To verify the effectiveness of our result, we consider the following discrete-time MJLS with $s = 3$:

$$A_1 = \begin{bmatrix} 0.25 & -0.83 \\ 2.50 & -3.50 \end{bmatrix}, A_2 = \begin{bmatrix} 1.0 & -0.25 \\ 2.5 & -3.00 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1.5 & -0.56 \\ 2.5 & -2.75 \end{bmatrix}, \ B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ B_3 = \begin{bmatrix} 0.8 \\ -1 \end{bmatrix},$$

$$\Pi^{(1)} = \begin{bmatrix} 0.0 & 0.5 & 0.5 \\ 0.3333 & 0.6667 & 0.0 \\ 0.5000 & 0.0 & 0.5 \end{bmatrix},$$

$$\Pi^{(2)} = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.0 & 0.3333 & 0.6667 \\ 0.25 & 0.5 & 0.25 \end{bmatrix},$$

$\lambda_{11} = 0.9091$, $\lambda_{12} = 0.0909$, $\lambda_{21} = 0.0833$, $\lambda_{22} = 0.9167$. By the definition of $\Pi^{(g)}$, the MTP $\pi_{ij}^{(g)}$ can be obtained by the (i, j)th element of $\Pi^{(g)}$. In addition, the control gains F_{gi} for multiple patterns (i.e., $c = 2$) can be characterized in terms of the solution to a set of LMIs in Theorem 1, which are given as follows:

$$F_{11} = \begin{bmatrix} 1.4229 & -1.8047 \end{bmatrix},$$
$$F_{21} = \begin{bmatrix} 1.4539 & -1.8532 \end{bmatrix},$$
$$F_{12} = \begin{bmatrix} -4.0191 & 5.7839 \end{bmatrix},$$
$$F_{22} = \begin{bmatrix} -3.8518 & 5.4816 \end{bmatrix},$$
$$F_{13} = \begin{bmatrix} 0.8039 & -1.3743 \end{bmatrix},$$
$$F_{23} = \begin{bmatrix} 1.1040 & -1.6588 \end{bmatrix}.$$

Fig. 3 shows the behavior of the state response by Algorithm 1 based on the obtained control gains, and the mode evolution used therein, where $x_0 = [-0.3 \;\; 0.4]^T$, $p_0 = 1$, and $r_0 = 1$. Here, by letting the cost index $\mathcal{J}_m = \sum_{k=0}^{m} x_k^T x_k$, it follows that $\mathcal{J}_{20} = 21.0661$ for $c = 1$ and $\mathcal{J}_{20} = 14.9114$ for $c = 2$. In this sense, we can see that, in comparison with the case of $c = 1$, Theorem 1 is in a better position for improving system performance because it offers the multiple pattern-dependent stabilization condition.

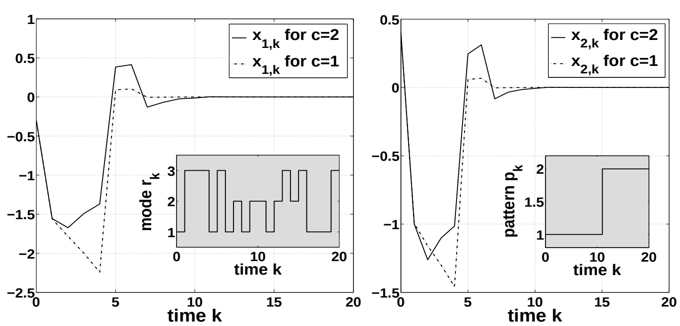

Fig. 3. Mode evolution and behavior of the state response $x_k = [x_{1,k} \; x_{2,k}]^T$.

VI. Concluding remarks

In this paper, we have paid considerable attention to deriving the multiple pattern-dependent stabilization condition for a class of discrete-time MJLSs. Our future work is directed toward extending our result to other interesting problems associated with MJLSs.

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