Necessary Conditions of Optimality for Stochastic Switching Systems With Delay

Charkaz Aghayeva

***Abstract*—** This paper provides necessary condition of optimality, in the form of maximum principle for optimal control problems of switching systems with constraints. Dynamics of the processes are defined by the stochastic differential equations with delay in the drift and diffusion coefficients. The restrictions on switches between operating mode are described by collection of functional constraints.

***Keywords*—** Optimal control problem, Stochastic differential equation with delay, Stochastic switching system, Switching law.

I. Introduction

THE stochastic differential equations with delay find much exhibits in description of the real systems, which in one or another degree are subjected to the influence of the random noises. Systems with stochastic uncertainties have provided a lot of interest for problems of nuclear fission, communication systems , self-oscillating systems and etc., where the influences of random disturbances can not be ignored [1]-[3]. Switching systems consist of several subsystems and a switching law indicating the active subsystem at each time instantly. Optimization problems for switching systems have attracted a lot of interest. Theoretical results and applications were developed in [4]-[6]. For general theory of stochastic switching systems it is referred to [7] .

Therefore problems of optimal control for switching systems, described by deterministic and stochastic differential equations with delay, are actual at present [8],[9]. Earlier the problems of stochastic optimal control of switching systems without delay were considered in [10]-[12]. Stochastic optimal control problem of unrestricted switching systems with delay is investigated in [13].

The present work is devoted to the optimal control problem of delayed stochastic switching system with uncontrolled diffusion coefficient. It is obtain maximum principle in the case when endpoint constraints are imposed. Using Ekeland's variational principle [14], given problem is convert into the sequence of unconstrained problems . Due to the result from [13 it is established maximum principle and transversality conditions. Finally, taking the limit, we achieve the necessary condition of optimality in the case when endpoint constraints are imposed.

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II. Preliminaries and Statement of Problem

Let $(\Omega, F^l, P), l=1,...,r$ be a probability spaces with filtration $\left\{ F_t^l, t\in[t_{l-1},t_l], l=1,...,r \right\}$, $0=t_0<t_1<...<t_r=T$. Assume that $w_t^1, w_t^2,...,w_t^r$ are independent Wiener processes, which generate $F_t^l=\overline{\sigma}(w_q^l, t_{l-1}\le t\le t_l)$, $l=1,...,r$. Let R^n denotes the n - dimensional real vector space and $|.|$ denotes the Euclidean norm in R^n . E represents the expectation. $L^2_{F^l}\left(a,b;R^n\right)$ denotes the space of all predictable processes $x_t(\omega)$ such that: $E\int_a^b \left|x_t(\omega)\right|^2 dt<+\infty$. $R^{m\times n}$ is the space of linear transformations from R^m to R^n. Let, $O_l\subset R^{n_l}, Q_l\subset R^{m_l}$ be open sets and $\mathrm{T}=[0,T]$ be a finite interval.

Consider the following stochastic control system with delay:

$$dx_t^l=g^l\left(x_t^l,x_{t-h}^l,u_t^l,t\right)dt+f^l\left(x_t,x_{t-h},t\right)dw_t^l \;\; t\in(t_{l-1},t_l] \quad (1)$$

$$x_t^{l+1}=\mathrm{K}^{l+1}(t), t\in[t_l-h,t_l), \; l=0,1,...,r-1, \quad (2)$$

$$x_{t_l}^{l+1}=\Phi^l\left(x_{t_l}^l,t_l\right),\; l=1,...,r-1,\;\; x_{t_0}^1=x_0, \quad (3)$$

$$u_t^l\in U_\partial^l\equiv\left\{u^l(\cdot,\cdot)\in L_F^2\left(t_{l-1},t_l;R^m\right) \mid u^l(t,\cdot)\in U^l\subset R^m\right\} \quad (4)$$

where U^l, $l=1,...,r$ are non-empty bounded sets.

Let $\Lambda_l, l=1,...,r$ be the set of piecewise continuous functions $\mathrm{K}^l(\cdot)$, $l=1,...,r$: $[t_{l-1}-h,t_{l-1})\to N_l\subset O_l$ and $h\ge 0$.

The problem is concluded to find the control $u^1,u^2,...,u^r$ and the switching law $t_1,t_2,...,t_r$ which minimize the cost functional :

$$J(u)=\sum_{l=1}^r E\left[\varphi^l\left(x_{t_l}^l\right)+\int_{t_{l-1}}^{t_l} p^l\left(x_t^l,u_t^l,t\right)dt\right] \quad (5)$$

which is determined on the decisions of the system (1)- (3), which are generated by all admissible controls $U=U^1\times U^2\times...\times U^r$ at conditions:

$$Eq^r\left(x_{t_r}^r\right)\in G \quad (6)$$

G is a closed convex set in R^k.

Consider the sets: $A_i = \mathrm{T}^{i+1} \times \prod_{j=1}^{i} O_j \times \prod_{j=1}^{i} \Lambda_j \times \prod_{j=1}^{i} U^j$ with the elements
$\pi^i = (t_0, t_1, t_i, x_{t_0}^1, x_{t_2}^2, ..., x_{t_i}^i, K_1, ..., K_i, u^1, u^2, ..., u^i)$.

Definition 1: The set of functions $\{x_t^l = x^l(t, \pi^l),\ t \in [t_{l-1} - h, t_l],\ l = 1, ...r\}$ is said to be a solution of the equation with variable structure which corresponds to an element $\pi^r \in \mathrm{A}_r$, if the function $x_t^l \in O_l$ on the interval $[t_l - h, t_l]$ satisfies the conditions (2),(3), while on the interval $[t_{l-1}, t_l]$ it is absolutely continuous with probability 1 and satisfies the equation (1) almost everywhere.

Definition 2: The element $\pi^r \in A_r$ is said to be admissible if the pairs $(x_t^l, u_t^l),\ t \in [t_{l-1} - h, t_l],\ l = 1, ..., r$ are the solutions of system (1)-(4) and satisfied the conditions (6) .
A_r^0 denotes the set of admissible elements.

Definition 3: The element $\tilde{\pi}^r \in A_r^0$, is said to be an optimal solution of problem (1)-(6) if there exist admissible controls $\tilde{u}_t^l, t \in [t_{l-1}, t_l], l = 1, ...r$ and corresponding solutions $\{\tilde{x}_t^l,\ t \in [t_{l-1} - h, t_l],\ l = 1, ..., r\}$ of system (1)-(4) with constraints (6), and pairs $(\tilde{x}_t^l, \tilde{u}_t^l),\ l = 1, ..., r$ minimize the functional (5).
Assume that the following requirements are satisfied:
I. Functions $g^l,\ f^l,\ p^l,\ l = 1, ..., r$ and their derivatives are continuous in (x, y, u, t):
$g^l(x, y, u, t) : O_l \times O_l \times Q_l \times T \to R^{n_l}$
$f^l(x, y, t) : O_l \times O_l \times T \to R^{n_l \times n_l}$ $p^l(x, u, t) : O_l \times Q_l \times T \to R^1$.
II. When (t, u) are fixed , functions $g^l,\ f^l,\ p^l,\ l = \overline{1, r}$ hold the conditions:
$$(1 + |x| + |y|)^{-1}\left(\left|g^l(x, y, u, t)\right| + \left|g_x^l(x, y, u, t)\right| + \left|g_y^l(x, y, u, t)\right| + \left|f^l(x, y, t)\right| + \left|f_x^l(x, y, t)\right| + \left|f_y^l(x, y, t)\right| + \left|p^l(x, u, t)\right| + \left|p_x^l(x, u, t)\right|\right) \le N.$$
III. Functions $\varphi^l(x) : R^{n_l} \to R^1, l = 1, ..., r$ are continuously differentiable and satisfies the following:
$$\left|\varphi^l(x)\right| + \left|\varphi_x^l(x)\right| \le N(1 + |x|) .$$
IV Functions $\Phi^l(x, t_l) : R^{n_l} \times \mathrm{T} \to R^1, l = 1, ..., r-1$ are continuously differentiable in respect to (x, t):
$$\left|\Phi^l(x, t_l)\right| + \left|\Phi_x^l(x, t_l)\right| \le N(1 + |x|) .$$
V. Functions $q^r(x) : R^{n_r} \to R$ are continuously differentiable in respect to (x, t) :
$$\left|q^l(x)\right| + \left|q_x^l(x)\right| \le N(1 + |x|) .$$

III. Methods of Solution

The following result that is a necessary condition of optimality for problem (1)-(5) has been obtained in [6].

Theorem 1 . Suppose that assumptions I-IV hold, $\pi^r = (t_0, .., t_r, x_t^1, ..., x_t^r, K_1, ..., K_r, u^1, ..., u^r)$ is an optimal solution of problem (1)-(5) and random processes $(\psi_t^l, \beta_t^l) \in L_{F^l}^2(t_{l-1}, t_l; R^{n_l}) \times L_{F^l}^2(t_{l-1}, t_l; R^{n_l x n_l})$ are the solutions of the following adjoint equations:
$$\begin{cases} d\psi_t^l = -\left[H_x^l(\psi_t^l, x_t^l, y_t^l, u_t^l, t) + H_y^l(\psi_{t+h}^l, x_{t+h}^l x_t^l, u_t^l, t)\right]dt \\ \qquad + \beta_t^l dw_t^l,\ \ t_{l-1} \le t < t_l - h, \\ d\psi_t^l = -H_x^l(\psi_t^l, x_t^l, y_t^l, u_t^l, t)dt + \beta_t^l dw_t^l,\ \ t_{l-1} - h_l \le t < t_l, \\ \psi_{t_l}^l = -\varphi_x^l(x_{t_l}^l) + \psi_{t_{l+1}}^l \Phi_x^l(x_{t_l}^l, t_l),\ l = 1, ..., r-1, \\ \psi_{t_r}^r = -\varphi^r(x_{t_r}^r). \end{cases}$$

Then ,
a) almost certainly for $\forall \tilde{u}^l \in U^l, l = 1, ...r,$ a.e. in $[t_{l-1}, t_l]$ the maximum principle hold:
$$H^l(\psi_\theta^l, x_\theta^l, y_\theta^l, \tilde{u}^l, \theta) - H^l(\psi_\theta^l, x_\theta^l, y_\theta^l, u_\theta^l, \theta) \le 0$$
b) Following transversality conditions hold:
$$-a_l\psi_{t_l}^l g^l(x_{t_l}^l, y_{t_l}^l, u_{t_l}^l, t_l) + b_l\psi_{t_l}^{l+1} g^{l+1}(x_{t_l}^l, y_{t_l}^l, u_{t_l}^l, t_l) + b_l\psi_{t_l+h}^{l+1} g^{l+1}(x_{t_l}^l, K^l(t_l), u_{t_l}^l, t_l) - b_l\Phi_t^l(x_{t_l}^l, t_l) = 0,\ \ l = 0, 1, ..., r$$
Here
$$H^l(\psi_t, x_t, y_t, u_t, t) = \psi_t g^l(x_t, y_t, u_t, t) + \beta_t f^l(x_t, y_t, t) - p^l(x_t, u_t, t),\ \ t \in [t_{l-1}, t_l],$$
$y_t^l = x_{t-t}^l$, $a_0 = 0, a_1 = ... = a_r = 1$ and
$b_0 = ... = b_{r-1} = 1, b_r = 0.$

Further, by applying Theorem 1 and Ekeland's Variational Principle it is obtained the necessary condition of optimality for stochastic control problem of switching systems with delay (1)-(6).

Theorem 2 . Suppose that, assumptions I-V hold, $\pi^r = (t_0, ..., t_r, x_t^1, x_t^2, ..., x_t^r, K_1, ..., K_r, u^1, u^2, ..., u^r)$ is a optimal solution of problem (1)-(6) and random processes $(\psi_t^l, \beta_t^l) \in L_{F^l}^2(t_{l-1}, t_l; R^{n_l}) \times L_{F^l}^2(t_{l-1}, t_l; R^{n_l x n_l})$ are the solutions of the following adjoint equations:
$$\begin{cases} d\psi_t^l = -\left[H_x^l(\psi_t^l, x_t^l, y_t^l, u_t^l, t) + H_y^l(\psi_{t+h}^l, x_{t+h}^l x_t^l, u_t^l, t)\right]dt \\ \qquad + \beta_t^l dw_t^l,\ \ t_{l-1} \le t < t_l - h, \\ d\psi_t^l = -H_x^l(\psi_t^l, x_t^l, y_t^l, u_t^l, t)dt + \beta_t^l dw_t^l,\ \ t_{l-1} - h \le t < t_l \\ \psi_{t_l}^l = -\lambda_l\varphi_x^l(x_{t_l}^l) + \psi_{t_{l+1}}^l \Phi_x^l(x_{t_l}^l, t_l),\ l = 1, ..., r-1 \\ \psi_{t_r}^r = -\lambda_0\varphi_x^r(x_{t_r}^r) - \lambda_r q_x^r(x_{t_r}^r). \end{cases} \qquad (7)$$
Then ,
a) almost certainly for $\forall \tilde{u}^l \in U^l, l = 1, ...r,$ a.e. in $[t_{l-1}, t_l]$ the maximum principle holds:
$$H^l(\psi_\theta^l, x_\theta^l, y_\theta^l, \tilde{u}^l, \theta) - H^l(\psi_\theta^l, x_\theta^l, y_\theta^l, u_\theta^l, \theta) \le 0 \qquad (8)$$
b) following transversality condition holds a.c.:
b) Following transversality conditions hold:

$$-a_l\psi^l_{t_l}g^l\left(x^l_{t_l},y^l_{t_l},u^l_{t_l},t_l\right)+b_l\psi^{l+1}_{t_l}g^{l+1}\left(x^l_{t_l},y^l_{t_l},u^l_{t_l},t_l\right)+$$
$$b_l\psi^{l+1}_{t_l+h}g^{l+1}\left(x^l_{t_l},K^l(t_l),u^l_{t_l},t_l\right)-b_l\Phi^l_t\left(x^l_{t_l},t_l\right)=0,\ \ l=0,1,...,r$$
(9)

where $a_0=0,a_1=....=a_r=1$ and $b_0=....=b_{r-1}=1,b_r=0.$

Proof. For any natural j let's introduce the approximating functional:

$$I_j(\mathbf{u})=S^l_j(\sum_{l=1}^r\left[E\varphi^l\left(x^l_{t_l}\right)+E\int_{t_{l-1}}^{t_l}p^l\left(x^l_t,u^l_t,t\right)dt\right],Eq^r(x^r_{t_r}))=$$
$$\min_{c^l_j\in\varepsilon}\sqrt{\sum_{l=1}^r\left|c^l_j-1/j-E\left[\varphi^l(x^l_{t_l})+\int_{t_{l-1}}^{t_l}p(x^l_t,u^l_t,t)dt\right]\right|^2+\left|y-Eq^r(x^r_{t_r})\right|^2}$$

Where $\varepsilon=\{c:c\le J^0,y\in G\}$ and J^0 is minimal value of the functional in the problem (1)-(5). Let $\mathbf{V}\equiv(V^1,...,V^r)$, here $V^k\equiv(U^k,d)$ be space of controls obtained by means of the following metric:

$d(u^k,v^k)=(l\otimes P)\{(t,\omega)\in[t_{k-1},t_k]\times\Omega:v^k_t\ne u^k_t\}.$

It is easy to prove the following fact:

Lemma 1. Assume that conditions I-IV hold, $u^{l,n}_t,l=1,...,r$ be the sequence of admissible controls from V^l, and $x^{l,n}_t$ be the sequence of corresponding trajectories of the system (1)-(3). If the following condition is met: $d(u^{l,n}_t,u^l_t)\to 0$.

Then

$$\lim_{n\to\infty}\left\{\sup_{t_{l-1}\le t\le t_l}E\left|x^{l,n}_t-x^l_t\right|^2\right\}=0$$

where x^l_t is a trajectory corresponding to an admissible controls $u^l_t,l=1,...,r$.

According to Ekeland's variational principle, there are controls such as; $u^{l,j}_t:d(u^{l,j}_t,u^l_t)\le\sqrt{\varepsilon^l_j}$ and for $\forall u^l_t\in V^l$ the following is achieved:

$$I_j(\mathbf{u}^j)\le I_j(\mathbf{u})+\sum_{l=1}^r\sqrt{\varepsilon^l_j}d(u^{l,j},u^l),\ \varepsilon^l_j=\frac{1}{j}.$$

This inequality means that $(t_0,t_1,...,t_r,x^{1,j}_t,...,x^{r,j}_t,K_1,...,K_r,u^{1,j}_t,...,u^{r,j}_t)$ is a solution of the following problem:

$$\begin{cases}J_j(\mathbf{u})=I_j(\mathbf{u}^j)+\sum_{l=1}^r\sqrt{\varepsilon^l_j}E\int_{t_{l-1}}^{t_l}\delta(u^l_t,u^{l,j}_t)dt\to\min\\ dx^l_t=g^l(x^l_t,y^l_t,u^l_t,t)dt+f^l(x^l_t,y^l_t,t)dw_t,\ t\in(t_{l-1},t_l]\\ x^{1+1}_t=\mathrm{K}^{l+1}(t),t\in[t_l-h,t_l),\ l=0,1,...,r-1,\\ x^{l+1}_{t_l}=\Phi^l\left(x^l_{t_l},t_l\right)\ \ l=1,...,r\\ x^1_{t_0}=x_0,\\ u^l_t\in U^l_\partial\end{cases}\qquad(10)$$

Function $\delta(u,v)$ is determined in the following way:

$$\delta(u,v)=\begin{cases}0,u=v\\1,u\ne v.\end{cases}$$

Then according to the Theorem 1, it is obtained as follows:

1) there exist the random processes $\psi^{l,j}_t\in L^2_{F^l}(t_{l-1},t_l;R^{n_l})$, $\beta^{l,j}_t\in L^2_{F^l}(t_{l-1},t_l;R^{n_l\times n_l})$, which are solutions of the following system

$$\begin{cases}d\psi^{l,j}_t=-H^l_x\left(\psi^{l,j}_t,x^{l,j}_t,y^{l,j}_t,u^{l,j}_t,t\right)dt-H^l_y\left(\psi^{l,j}_{t+h},x^{l,j}_{t+h},y^{l,j}_{t+h},u^{l,j}_{t+h},t+h\right)dt\\ +\beta^{l,j}_tdw_t,\ t\in[t_{l-1},t_l-h),\ l=1,...,r\\ d\psi^{l,j}_t=-H^l_x\left(\psi^{l,j}_t,x^{l,j}_t,y^{l,j}_t,u^{l,j}_t,t\right)dt+\beta^{l,j}_tdw_t,\ t\in[t_{l-1}-h,t_l),\\ \psi^{l,j}_{t_l}=-\lambda^j_0\varphi^l_x\left(x^{l,j}_{t_l}\right)+\psi^l_{t_{l+1}}\Phi^l_x(x^{l,j}_{t_l},t_l),\ l=1,...,r-1\\ \psi^r_{t_r}=-\lambda^j_0\varphi^r_x\left(x^{r,j}_{t_r}\right)-\lambda^j_rq^r_x\left(x^{r,j}_{t_r}\right).\end{cases}$$

(11)

where non-zero $(\lambda^j_0,\lambda^j_1,...,\lambda^j_r)\in R^{r+1}$ meet the following requirement:

$$\lambda^j_l=\left(-c_l+1/j+E\varphi^l\left(x^{l,j}_{t_l}\right)+E\int_{t_{l-1}}^{t_l}p^l(x^{l,j}_t,u^{l,j}_t,t)dt\right)/J^0_j$$
$$\lambda^j_r=Eq^r\left(x^{r,j}_{t_r}\right)/J^0_j$$
$$J^0_j=\sqrt{\sum_{l=1}^r\left|c_l-1/j-E\left[\varphi^l(x^l_{t_l})+\int_{t_{l-1}}^{t_l}p(x^l_t,u^l_t,t)dt\right]\right|^2+\left|y-Eq^r(x^r_{t_r})\right|^2}$$

2) almost certainly for any $\tilde{u}^l\in U^l$ and a.e. $t\in[t_{l-1},t_l]$ is satisfied:

$$H^l\left(\psi^{l,j}_t,x^{l,j}_t,y^{l,j}_t,\tilde{u}^l_t,t\right)-H^l\left(\psi^{l,j}_t,x^{l,j}_t,y^{l,j}_t,u^{l,j}_t,t\right)\le 0\quad(12)$$

3) the following transversality conditions hold:

$$-a_l\psi^{l,j}_{t_l}g^l\left(x^{l,j}_{t_l},y^{l,j}_{t_l},u^{l,j}_{t_l},t_l\right)+b_l\psi^{l+1,j}_{t_l}g^{l+1}\left(x^{l,j}_{t_l},y^{l,j}_{t_l},u^{l,j}_{t_l},t_l\right)+$$
$$b_l\psi^{l+1,j}_{t_l+h}g^{l+1}\left(x^{l,j}_{t_l},K^{l,j}(t_l),u^{l,j}_{t_l},t_l\right)-b_l\Phi^l_t\left(x^{l,j}_{t_l},t_l\right)=0,\ \ l=0,1,...,r$$
(13).

Since the following exists $\left|(\lambda^j_0,\lambda^j_1,...,\lambda^j_r)\right|=1$, then according to conditions I-IV it is implied that

$$(\lambda^j_0,\lambda^j_1,...,\lambda^j_r)\to(\lambda_0,\lambda_1,...,\lambda_r)\ \text{if}\quad j\to\infty.$$

Let us introduce the following result which will be needed in the future.

Lemma 2. Let $\psi^l_{t_l}$ be a solution of system (7), and $\psi^{l,j}_{t_l}$ be a solution of system (11). Then

$$E\int_{t_{l-1}}^{t_l}|\psi_t^{l,j}-\psi_t^l|^2\,dt+E\int_{t_{l-1}}^{t_l}|\beta_t^{l,j}-\beta_t^l|^2\,dt\to 0,\text{ if}$$

$d(u_t^{l,j},u_t^l)\to 0,\ j\to\infty$.

Proof: It is clear that $\forall t\in[t_{l-1},t_l],l=1,...,r-1$:

$$d\left(\psi_t^{l,j}-\psi_t^l\right)=-\left[H_x^l\left(\psi_t^{l,j},x_t^{l,j},y_t^{l,j},u_t^{l,j},t\right)-H_x^l\left(\psi_t^l,x_t^l,y_t^l,u_t^l,t\right)\right]dt+\left(\beta_t^{l,j}-\beta_t^l\right)dw_t$$

According to Ito formula, for $\forall s\in[t_l-h,t_l]$ it is satisfied:

$$E|\psi_{t_l}^{l,j}-\psi_{t_l}^l|^2-E|\psi_s^{l,j}-\psi_s^l|^2=$$
$$2E\int_s^{t_r}[\psi_t^{l,j}-\psi_t^l][(g_x^{l*}(x_t^{l,j},y_t^{l,j},u_t^{l,j},t)-g_x^{l*}(x_t^l,y_t^l,u_t^l,t))\psi_t^{l,j}+$$
$$g_x^{l*}(x_t^l,y_t^l,u_t^l,t)(\psi_t^{l,j}-\psi_t^l)+(f_x^{l*}(x_t^{l,j},y_t^{l,j},t)-f_x^{l*}(x_t^l,y_t^l,t))\beta_t^{l,j}$$
$$-p^l(x_t^{l,j},u_t^{l,j},t)+p_x^l(x_t^l,u_t^l,t)]dt+E\int_s^{t_r}|\beta_t^{l,j}-\beta_t^l|^2\,dt.$$

Due to assumptions I-IV and using simple transformations, the following is obtained:

$$E\int_s^{t_l}|\beta_t^{l,j}-\beta_t^l|^2\,dt+E|\psi_s^{l,j}-\psi_s^l|^2\le EN\int_s^{t_l}|\psi_t^{l,j}-\psi_t^l|^2\,dt+$$
$$+EN\varepsilon\int_s^{t_r}|\beta_t^{l,j}-\beta_t^l|^2\,dt+E\left|\psi_{t_l}^{l,j}-\psi_{t_l}^l\right|^2.$$

Hence, according to Gronwall inequality [3] it suggests that:

$$E|\psi_s^{l,j}-\psi_s^l|^2\le De^{N(t_r-s)}\quad\text{a.e. in }\ [t_l-h,t_l]$$

(14)

where constant D is determined in the way below:

$D=E|\psi_{t_l}^{l,j}-\psi_{t_l}^l|^2$.

According to (7) and (11), it is obtained that: $\psi_{t_l}^{l,j}\to\psi_{t_l}^l$,

which leads to $D\to 0$. Consequently, from (14) it follows:

$\psi_s^{l,j}\to\psi_s^l$ in $L_{F^l}^2(t_l-h,t_l;R^{n_l})$ and $\beta_s^{l,j}\to\beta_s^l$

in $L_{F^l}^2(t_l-h,t_l;R^{n_l\times n_l})$.

Then, $\forall t\in\left[t_{l-1},t_l-h\right),l=1,...,r$ from the expression:

$$d\left(\psi_t^{l,j}-\psi_t^l\right)=-\left[H_x^l\left(\psi_t^{l,j},x_t^{l,j},y_t^{l,j},u_t^{l,j},t\right)-H_x^l\left(\psi_t^l,x_t^l,y_t^l,u_t^l,t\right)\right]dt$$
$$-\left[H_y^l\left(\psi_{t+h}^{l,j},x_{t+h}^{l,j},y_{t+h}^{l,j},u_{t+h}^{l,j},t+h\right)-H_y^l\left(\psi_{t+h}^l,x_{t+h}^l,y_{t+h}^l,u_{t+h}^l,t+h\right)\right]dt$$
$$+\left(\beta_t^{l,j}-\beta_t^l\right)dw_t$$

using simple transformations , in view of assumptions I-IV the following is obtained:

$$E\int_s^{t_l-h}|\beta_t^{l,j}-\beta_t^l|^2\,dt+E|\psi_s^{l,j}-\psi_s^l|^2\le EN\int_s^{t_l-h}\left|\psi_t^{l,j}-\psi_t^l\right|^2dt+$$
$$+EN\varepsilon\int_s^{t_l-h}|\beta_t^{l,j}-\beta_t^l|^2\,dt+E|\psi_{t_l-h}^{l,j}-\psi_{t_l-h}^l|^2.$$

Hence, according to Gronwall inequality, the following result is achieved:

$$E|\psi_s^{l,j}-\psi_s^l|^2\le De^{N(t_l-s)}\quad\text{a.e. in }\ \left[t_{l-1},t_l-h\right)$$

where constant D is determined as follows:

$D=E|\psi_{t_l-h}^{l,j}-\psi_{t_l-h}^l|^2$, which leads to $D\to 0$.

It is inferred that $\psi_s^{l,j}\to\psi_s^l$ in $L_{F^l}^2(t_{l-1},t_l;R^{n_l})$ and $\beta_s^{l,j}\to\beta_s^l$ in $L_{F^l}^2(t_{l-1},t_l;R^{n_l\times n_l})$.

Lemma 2 is proved.

It follows from Lemma 2 that it can be proceeded to the limit in system (11) and the fulfilments of (7) are obtained. Following the similar scheme by taking limit in (12) and (13) it is proved that (8), (9) are true. Theorem 2 is proved.

IV. CONCLUSION

It is obtained a necessary condition of optimality for stochastic control problem of switching systems with delay on state. Necessary conditions satisfied by an optimal solution , play an important role for investigation of optimal control problems. The result can be used in various optimal control problems of biological, technical and economic systems. The necessary conditions developed in this study can be viewed as a stochastic analogues of the problems formulated in ([4]-[6]). However, Theorem 2 is a natural evolution of the results given in [10]-[13].

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