Gaussian type differential equation

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***Abstract*—Lie point symmetries and a new approach utilizing Euler's type formulas for the solution of second order ordinary linear differential equations are applied to determine symmetries for a differential equation derived from a Gaussian function whose antiderivate cannot be expressed in closed form. The effectiveness of the approach is tested by constructing invariant solutions of the symmetries if any.**

***Index Terms*—Gaussian function, invariant solution, partial differential equation, point symmetries.**

I. INTRODUCTION

THE Gaussian function

$$\int_0^{\infty} e^{-ax^2}dx \tag{1}$$

is classified as an integral whose antiderivative cannot be expressed in closed form (i.e. cannot be expressed analytically in terms of a finite number of certain well known functions) [4].

The current undertaking seeks to determine the solution of its derived differential equation using Lie Symmetry method. Lie Symmetry method is a mathematical theory that synthesizes symmetry of differential equation [2].

In order to apply Lie Symmetry method to the Gaussian type function, we need to first present it as a differential equation by substituting

$$a = t,$$

and letting

$$u = \int_0^{\infty} e^{-tx^2}dx \tag{2}$$

resulting in

$$u_x = -2txe^{-tx^2}. \tag{3}$$

If we differentiate equation (3) with respect to t then the resulting partial differential equation becomes

$$u_{tx} = \frac{1}{t}u_x - x^2u_x. \tag{4}$$

Equation (4) is a partial differential equation with independent variables t and x, and differential variable u.

II. SOLUTION OF DETERMINING EQUATION

The infinitesimal generator for point symmetry admitted by equation (4) is of the form

$$X = \xi^1(t,x)\frac{\partial}{\partial t} + \xi^2(t,x)\frac{\partial}{\partial x} + \eta(t,x)\frac{\partial}{\partial u} \tag{5}$$

Its first and second prolongations are given by

$$X^{(2)} = X + \eta_x^{(1)}\frac{\partial}{\partial u_x} + \eta_{tx}^{(2)}\frac{\partial}{\partial u_{tx}} \tag{6}$$

where X is defined by equation (5). The invariance condition for (4) is given by

$$\begin{aligned}&X^{(2)}(u_{tx} - \frac{1}{t}u_x + x^2u_x)|_{u_{tx}=\frac{1}{t}u_x - x^2u_x} = \\ &(\eta_{tx}^{(2)} - \frac{1}{t}\eta_x^{(1)} + \frac{1}{t^2}\xi^1u_x + 2x\xi^{(2)}u_x + x^2\eta_x^{(1)})|_{u_{tx}=\frac{1}{t}u_x - x^2u_x} = 0\end{aligned} \tag{7}$$

We define the following from ([1],[2])

$$\begin{aligned}\eta &= fu + g\\ \eta_t^{(1)} &= g_t + f_tu + [f - \xi_t^1]u_t - \xi_t^2u_x\\ \eta_x^{(1)} &= g_x + f_xu + [f - \xi_x^2]u_x - \xi_x^1u_t\\ \eta_{tx}^{(2)} &= g_{tx} + f_{tx}u + [f_t - \xi_{tx}^2]u_x + [f_x - \xi_{tx}^1]u_t\\ &\quad + u_{tx}[f - \xi_t^1 - \xi_x^2] - \xi_t^2u_{xx} - \xi_x^1u_{tt}\end{aligned} \tag{8}$$

The substitutions of $\eta_x^{(1)}$ and $\eta_{tx}^{(2)}$ in the invariance condition (7) yield the determining equation

$$\begin{aligned}&g_{tx} + f_{tx}u + [f_t - \xi_{tx}^2]u_x + [f_x - \xi_{tx}^1]u_t + (\frac{1}{t}u_x - x^2u_x)[f - \xi_t^1 - \xi_x^2]\\ &- \xi_t^2u_{xx} - \frac{1}{t}g_x - \frac{1}{t}f_xu - \frac{1}{t}u_x[f - \xi_x^2] + \frac{1}{t}u_t\xi_x^1 + \frac{1}{t^2}\xi^1u_x - \xi_x^1u_{tt}\\ &+ 2x\xi^2u_x + x^2g_x + x^2f_xu + x^2u_x[f - \xi_x^2] - x^2\xi_x^1u_t = 0\end{aligned} \tag{9}$$

We set the coefficients of $u_{xx}, u_{tt}, u_x, u_t, u$ and those free of these variables to zero. We thus have the following defining equations

$$u_{xx} \quad : \quad \xi_t^2 = 0, \tag{10}$$

$$u_{tt} \quad : \quad \xi_x^1 = 0, \tag{11}$$

$$u_t \quad : \quad f_x = 0, \tag{12}$$

$$u_x \quad : \quad f_t - \frac{1}{t}\xi_t^1 + \frac{1}{t^2}\xi^1 + 2x\xi^2 + x^2\xi_t^1 = 0, \tag{13}$$

$$u \quad : \quad f_{tx} = 0, \tag{14}$$

$$u^0 \quad : \quad g_{tx} = \frac{1}{t}g_x - x^2g_x. \tag{15}$$

We differentiate defining equation (13) with respect to t and apply equation (10) to obtain the equation

$$f_{tt} - (\frac{1}{t}\xi_t^1)_t + (\frac{1}{t}\xi^1)_t + x^2\xi_{tt}^1 = 0 \tag{16}$$

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The derivative of equation (16) with respect to x and the application of equations (11) and (12) result in that

$$2x\xi^1_{tt} = 0$$

whence

$$\xi^1_{tt} = 0 \tag{17}$$

Thus we have that

$$\xi^1 = at + b \tag{18}$$

which can be expressed using Euler formula with infinitesimal ω as

$$\xi^1 = \frac{a\ \sin(\frac{\omega t}{i}) + b\phi\ \cos(\frac{\omega t}{i})}{-i\omega} \\ \text{where} \quad \phi = \sin(\frac{\omega}{i}) \quad \text{and} \quad a = a(x), b = b(x). \tag{19}$$

We differentiate equation (19) with respect to t and obtain expressions for ξ^1_t, and ξ^1_{tt}

$$\xi^1_t = a\ \cos(\frac{\omega t}{i}) - b\phi\ \sin(\frac{\omega t}{i}), \tag{20}$$

$$\xi^1_{tt} = \frac{-\omega}{i}\ a\ \sin(\frac{\omega t}{i}) - \frac{\omega}{i}\ b\ \phi\ \cos(\frac{\omega t}{i}), \tag{21}$$

Similarly we differentiate defining equation (13) with respect to x and obtain

$$(x\xi^2)_x = -x\xi^1_t \tag{22}$$

We integrate equation (22) with respect to x and simplify to obtain the expression for ξ^2, given as

$$\xi^2 = -\frac{1}{2}x\xi^1_t + A \tag{23}$$

which translate to

$$\xi^2 = -\frac{1}{2}ax\ \cos(\frac{\omega t}{i}) + \frac{1}{2}bx\phi\ \sin(\frac{\omega t}{i}) + A. \tag{24}$$

The equation (16) imply that

$$f_{tt} = (\frac{1}{t}\xi^1_t)_t - (\frac{1}{t}\xi^1)_t - x^2\xi^1_{tt}$$

which translate to

$$\begin{aligned} f_{tt} = &-\frac{\omega a}{it}\ \sin(\frac{\omega t}{i}) - \frac{\omega b}{it}\phi\ \cos(\frac{\omega t}{i}) - \frac{a}{t^2}\cos(\frac{\omega t}{i}) \\ &+ \frac{b\phi}{t^2}\sin(\frac{\omega t}{i}) - \frac{a}{t}\ \cos(\frac{\omega t}{i}) + \frac{b\phi}{t}\ \sin(\frac{\omega t}{i}) \\ &- \frac{a}{it^2\omega}\ \sin(\frac{\omega t}{i}) - \frac{b\phi}{it^2\omega}\ \cos(\frac{\omega t}{i}) + \frac{ax^2\omega}{i}\ \sin(\frac{\omega t}{i}) \\ &+ \frac{b\phi x^2\omega}{i}\ \cos(\frac{\omega t}{i}) \end{aligned} \tag{25}$$

The integration of equation (25) results in the expression for f_t and f given as

$$\begin{aligned} f_t = &\frac{a}{t}\ \cos(\frac{\omega t}{i}) - \frac{b}{t}\phi\ \sin(\frac{\omega t}{i}) - \frac{ai}{\omega t^2}\sin(\frac{\omega t}{i}) \\ &- \frac{bi\phi}{\omega t^2}\cos(\frac{\omega t}{i}) - \frac{ia}{\omega t}\ \sin(\frac{\omega t}{i}) - \frac{ib\phi}{t\omega}\ \cos(\frac{\omega t}{i}) \\ &+ \frac{a}{\omega^2 t^2}\ \cos(\frac{\omega t}{i}) - \frac{b\phi}{\omega^2 t^2}\ \sin(\frac{\omega t}{i}) - ax^2\ \cos(\frac{\omega t}{i}) \\ &+ b\phi x^2\ \sin(\frac{\omega t}{i}) \end{aligned} \tag{26}$$

and

$$\begin{aligned} f = &\frac{ai}{t\omega}\ \sin(\frac{\omega t}{i}) + \frac{ib}{t\omega}\phi\ \cos(\frac{\omega t}{i}) - \frac{a}{\omega^2 t^2}\cos(\frac{\omega t}{i}) \\ &+ \frac{b\phi}{\omega^2 t^2}\sin(\frac{\omega t}{i}) - \frac{a}{\omega^2 t}\ \sin(\frac{\omega t}{i}) - \frac{ib\phi}{t\omega}\ \sin(\frac{\omega t}{i}) \\ &+ \frac{ia}{\omega^3 t^2}\ \sin(\frac{\omega t}{i}) + \frac{ib\phi}{\omega^3 t^2}\ \cos(\frac{\omega t}{i}) - \frac{iax^2}{\omega}\ \sin(\frac{\omega t}{i}) \\ &- \frac{bi\phi x^2}{\omega}\ \cos(\frac{\omega t}{i}) + B \end{aligned} \tag{27}$$

respectively. From the defining equation (11) we have that

$$\xi^1_x = \frac{\dot{a}\ \sin(\frac{\omega t}{i}) + \dot{b}\phi\ \cos(\frac{\omega t}{i})}{-i\omega} = 0 \tag{28}$$

This result in that

$$\begin{gathered} \dot{a} = 0 \quad \text{and} \quad \dot{b} = 0 \\ \text{Hence} \quad a = C_1 \quad \text{and} \quad b = C_2 \end{gathered} \tag{29}$$

The defining equation (12) $f_x = 0$ imply that the last terms of f i.e.

$$-\frac{iax^2}{\omega}\ \sin(\frac{\omega t}{i}) - \frac{bi\phi x^2}{\omega}\ \cos(\frac{\omega t}{i}) = 0 \tag{30}$$

A. *Infinitesimals*

The linearly independent solutions of the defining equations (10) to (15) result in the infinitesimals

$$\xi^1 = -C_1\frac{\sin(\frac{\omega t}{i})}{i\omega} - C_2\phi\frac{\cos(\frac{\omega t}{i})}{i\omega} \tag{31}$$

$$\xi^2 = -\frac{1}{2}xC_1\cos(\frac{\omega t}{i}) + \frac{1}{2}x\phi C_2\sin(\frac{\omega t}{i}) + A \tag{32}$$

$$\begin{aligned} f = &\frac{C_1 i}{t\omega}\ \sin(\frac{\omega t}{i}) + \frac{iC_2}{t\omega}\phi\ \cos(\frac{\omega t}{i}) - \frac{C_1}{\omega^2 t^2}\cos(\frac{\omega t}{i}) \\ &+ \frac{C_2\phi}{\omega^2 t^2}\sin(\frac{\omega t}{i}) - \frac{C_1}{\omega^2 t}\ \sin(\frac{\omega t}{i}) - \frac{iC_2\phi}{t\omega}\ \sin(\frac{\omega t}{i}) \\ &+ \frac{iC_1}{\omega^3 t^2}\ \sin(\frac{\omega t}{i}) + \frac{iC_2\phi}{\omega^3 t^2}\ \cos(\frac{\omega t}{i}) + B \end{aligned} \tag{33}$$

B. *Symmetries*

The Symmetries according to infinitesimals (31) to (33) are:

$$\begin{aligned} X_1 = &\frac{\sin(\frac{\omega t}{i})}{i\omega}\frac{\partial}{\partial t} - \frac{1}{2}x\cos(\frac{\omega t}{i})\frac{\partial}{\partial x} \\ &\{ + \frac{i}{t\omega}\ \sin(\frac{\omega t}{i}) - \frac{1}{\omega^2 t^2}\cos(\frac{\omega t}{i}) \\ &- \frac{1}{\omega^2 t}\ \sin(\frac{\omega t}{i}) + \frac{i}{\omega^3 t^2}\ \sin(\frac{\omega t}{i})\}u\frac{\partial}{\partial u} \end{aligned} \tag{34}$$

$$\begin{aligned} X_2 = &-\phi\frac{\cos(\frac{\omega t}{i})}{i\omega}\frac{\partial}{\partial t} + \frac{1}{2}x\phi\sin(\frac{\omega t}{i})\frac{\partial}{\partial x} \\ &\{ + \frac{i}{t\omega}\phi\ \cos(\frac{\omega t}{i}) + \frac{\phi}{\omega^2 t^2}\sin(\frac{\omega t}{i}) \\ &- \frac{i\phi}{t\omega}\ \sin(\frac{\omega t}{i}) + \frac{i\phi}{\omega^3 t^2}\ \cos(\frac{\omega t}{i})\}u\frac{\partial}{\partial u} \end{aligned} \tag{35}$$

$$X_3 = \frac{\partial}{\partial x} \tag{36}$$

$$X_4 = u\frac{\partial}{\partial u} \tag{37}$$

The function $g(t,x)$ could not be determined and thus lead to an infinite symmetry generator

$$X_\infty = g(t,x)u\frac{\partial}{\partial u} \tag{38}$$

III. INVARIANT SOLUTIONS

A. Invariant solution through the symmetry X_2

We consider the symmetry given by equation (32). The invariants are determined from solving the equation

$$\begin{aligned} X_2 I = & -\phi\frac{\cos(\frac{\omega t}{i})}{i\omega}\frac{\partial I}{\partial t} + \frac{1}{2}x\phi\sin(\frac{\omega t}{i})\frac{\partial I}{\partial x} \\ & \{ + \frac{i}{t\omega}\phi\ \cos(\frac{\omega t}{i}) + \frac{\phi}{\omega^2 t^2}\sin(\frac{\omega t}{i}) \\ & - \frac{i\phi}{t\omega}\ \sin(\frac{\omega t}{i}) + \frac{i\phi}{\omega^3 t^2}\ \cos(\frac{\omega t}{i})\}u\frac{\partial I}{\partial u} = 0 \end{aligned} \tag{39}$$

The characteristic equation of (39) is given by

$$\begin{aligned} & -\frac{dt}{\phi\frac{\cos(\frac{\omega t}{i})}{i\omega}} = \frac{dx}{\frac{1}{2}x\phi\sin(\frac{\omega t}{i})} = \\ & \frac{du}{u\{\frac{i}{t\omega}\phi\ \cos(\frac{\omega t}{i}) + \frac{\phi}{\omega^2 t^2}\sin(\frac{\omega t}{i}) - \frac{i\phi}{t\omega}\ \sin(\frac{\omega t}{i}) + \frac{i\phi}{\omega^3 t^2}\ \cos(\frac{\omega t}{i})\}} \end{aligned} \tag{40}$$

From equation (40) we have that

$$-\frac{dt}{\phi\frac{\cos(\frac{\omega t}{i})}{i\omega}} = \frac{dx}{\frac{1}{2}x\phi\sin(\frac{\omega t}{i})} \tag{41}$$

simplifies to

$$\frac{2}{x}dx = -\omega i\tan(\frac{\omega t}{i})dt \tag{42}$$

The solution to equation (42) is given by

$$C + 2\ln x = -\ln\cos|(\frac{\omega t}{i})| \tag{43}$$

which result in that the first invariant is given by

$$C_1 = x^2\cos(\frac{\omega t}{i}) \tag{44}$$

Also from equation (40) we have that

$$\begin{aligned} & -i\omega\frac{dt}{\phi\cos(\frac{\omega t}{i})}\{\frac{i}{t\omega}\phi\ \cos(\frac{\omega t}{i}) + \frac{\phi}{\omega^2 t^2}\sin(\frac{\omega t}{i}) \\ & -\frac{i\phi}{t\omega}\ \sin(\frac{\omega t}{i}) + \frac{i\phi}{\omega^3 t^2}\ \cos(\frac{\omega t}{i})\} \\ & = \frac{du}{u} \end{aligned} \tag{45}$$

We simplify left hand side of equation (45) by multiplying through by $\frac{-i\omega}{\cos(\frac{\omega t}{i})}$, and for smaller value of ω we have the approximation

$$dt\{\frac{1}{t} - \frac{1}{t^2} - 0 + \frac{1}{t^2}\} = \frac{du}{u}$$

Hence the equation becomes

$$\frac{dt}{t} = \frac{du}{u} \tag{46}$$

The solution to equation (46) is

$$\frac{u}{t} = C_2 \tag{47}$$

Since C_1 is independent of u, every invariant solution is of the form

$$\frac{u}{t} = F(x^2\cos(\frac{\omega t}{i})) \tag{48}$$

or equivalently

$$u = tF(x^2\cos(\frac{\omega t}{i})) \tag{49}$$

Differentiating equation (49) we obtain

$$u_x = 2xtF'\cos(\frac{\omega t}{i}) \tag{50}$$

$$u_{xt} = 2xF'\cos(\frac{\omega t}{i}) - 2xt\frac{\omega}{i}F'\sin(\frac{\omega t}{i}) - x^3t\frac{\omega}{i}F''\sin(\frac{2\omega t}{i}) \tag{51}$$

We substitute for equations (50) and (51) in equation (4) and obtain

$$\begin{aligned} & 2xF'\cos(\frac{\omega t}{i}) - 2xt\frac{\omega}{i}F'\sin(\frac{\omega t}{i}) - x^3t\frac{\omega}{i}F''\sin(\frac{2\omega t}{i}) \\ & -2xF'\cos(\frac{\omega t}{i}) + 2x^3tF'\cos(\frac{\omega t}{i}) = 0 \end{aligned} \tag{52}$$

If we let $\omega \to 0$ equation (52) simplifies to

$$2x^3tF' = 0$$

or

$$F' = 0 \tag{53}$$

Hence

$$F = A \tag{54}$$

The solution is given by

$$u = At \tag{55}$$

where A is a constant.

B. Invariant solution through the symmetry X_1

We consider the symmetry given by equation (31). The invariants are determined from solving the equation

$$\begin{aligned} X_1 I = & -\frac{\sin(\frac{\omega t}{i})}{i\omega}\frac{\partial I}{\partial t} - \frac{1}{2}x\cos(\frac{\omega t}{i})\frac{\partial I}{\partial x} \\ & \{ + \frac{i}{t\omega}\ \sin(\frac{\omega t}{i}) - \frac{1}{\omega^2 t^2}\cos(\frac{\omega t}{i}) \\ & - \frac{1}{\omega^2 t}\ \sin(\frac{\omega t}{i}) + \frac{i}{\omega^3 t^2}\ \sin(\frac{\omega t}{i})\}u\frac{\partial I}{\partial u} = 0 \end{aligned} \tag{56}$$

The characteristic equation of (56) is given by

$$\begin{aligned} & -\frac{dt}{\frac{\sin(\frac{\omega t}{i})}{i\omega}} = \frac{dx}{-\frac{1}{2}x\cos(\frac{\omega t}{i})} \\ & = \frac{du}{u\{\frac{i}{t\omega}\ \sin(\frac{\omega t}{i}) - \frac{1}{\omega^2 t^2}\cos(\frac{\omega t}{i}) - \frac{1}{\omega^2 t}\ \sin(\frac{\omega t}{i}) + \frac{i}{\omega^3 t^2}\ \sin(\frac{\omega t}{i})\}} \end{aligned} \tag{57}$$

From equation (57) we have that

$$-\frac{dt}{\frac{\sin(\frac{\omega t}{i})}{i\omega}} = \frac{dx}{-\frac{1}{2}x\cos(\frac{\omega t}{i})} \tag{58}$$

simplifies to

$$\frac{2}{x}dx = \omega i\cot(\frac{\omega t}{i})dt \tag{59}$$

The solution to equation (59) is given by

$$A+2\ln x=-\ln\sin|(\frac{\omega t}{i})| \tag{60}$$

which result in that the first invariant is given by

$$A_1=x^2\sin(\frac{\omega t}{i}) \tag{61}$$

Also from equation (57) we have that

$$\begin{aligned}&-i\omega\frac{dt}{\sin(\frac{\omega t}{i})}\{\frac{i}{t\omega}\ \sin(\frac{\omega t}{i})-\frac{1}{\omega^2t^2}\cos(\frac{\omega t}{i})\\&-\frac{1}{\omega^2t}\ \sin(\frac{\omega t}{i})+\frac{i}{\omega^3t^2}\ \sin(\frac{\omega t}{i})\}\\&=\frac{du}{u}\end{aligned} \tag{62}$$

We simplify left hand side of equation (62) by multiplying through by $\frac{-i\omega}{\sin(\frac{\omega t}{i})}$, and for smaller value of ω we have the approximation

$$dt\{\frac{1}{t}-\frac{1}{t^2}-0+\frac{1}{t^2}\}=\frac{du}{u}$$

Hence the equation becomes

$$\frac{dt}{t}=\frac{du}{u} \tag{63}$$

The solution to equation (63) is

$$\frac{u}{t}=A_2 \tag{64}$$

Since A_1 is independent of u, every invariant solution is of the form

$$\frac{u}{t}=F(x^2\sin(\frac{\omega t}{i})) \tag{65}$$

or equivalently

$$u=tF(x^2\sin(\frac{\omega t}{i})) \tag{66}$$

Differentiating equation (66) we obtain

$$u_x=2xtF'\sin(\frac{\omega t}{i}) \tag{67}$$

$$\begin{aligned}u_{xt}&=2xF'\sin(\frac{\omega t}{i})+2xt\frac{\omega}{i}F'\cos(\frac{\omega t}{i})\\&+x^3t\frac{\omega}{i}F''\sin(\frac{2\omega t}{i})\end{aligned} \tag{68}$$

We substitute for equations (67) and (68) in equation (4) and obtain

$$\begin{aligned}&2xF'\sin(\frac{\omega t}{i})+2xt\frac{\omega}{i}F'\cos(\frac{\omega t}{i})+x^3t\frac{\omega}{i}F''\sin(\frac{2\omega t}{i})\\&-2xF'\sin(\frac{\omega t}{i})+2x^3tF'\sin(\frac{\omega t}{i})=0\end{aligned} \tag{69}$$

If we let $\omega\to 0$ in equation (69) we get no solution.

C. *Invariant solution through the symmetry X_3*

The invariant solution through symmetry $X_3=\frac{\partial}{\partial x}$ yields that

$$u=H(t) \tag{70}$$

where $H(t)$ denotes some function of t, consistent with equation (71)

D. *Conclusion*

The approach produced symmetries which provided a linear invariant solutions. This is consistent with the result in [4] that

$$\int_0^\infty e^{-tx^2}dx=\frac{1}{2}\sqrt{\frac{\pi}{t}},\quad t>0 \tag{71}$$

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